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NOTE

CONTRACTIBLE EDGES IN NON-SEPARATING CYCLES

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Dedicated to Professor Toshiro Tsuzuku on his sixtieth birthday

An edge of a k-connected graph is said to be k-contractible if the contraction of the edge results in a k-connected graph. We prove that every triangle-free k-connected graph G has an induced cycle C such that all edges of C are k-contractible and such that $G - V(C)$ is $(k-3)$-connected $(k \geq 4)$. This result unifies two theorems by Thomassen [5] and Egawa et. al. [3].

In this paper, we consider finite simple graphs. An edge e of a k-connected graph is said to be ***k*-contractible** if the contraction of e results in a k-connected graph. For $k \leq 3$, a k-connected graph of order at least $k+2$ has a k-contractible edge. On the other hand, Thomassen [5] remarked that for $k \geq 4$ there exist infinitely many k-connected graphs which do not have a k-contractible edge. However, in the same paper [5] Thomassen showed that a triangle-free k-connected graph has a k-contractible edge, and Egawa, Enomoto and Saito [3] studied the distribution of k-contractible edges in such a graph. In particular, they proved the following theorem.

Theorem A (Egawa, Enomoto and Saito [3]). *Let G be a k-connected graph. Then G has an induced cycle C such that every edge of C is k-contractible in G. (An induced cycle is a cycle which is also an induced subgraph.)* ∎

Recently, Dean [2] obtained the following stronger result.

Theorem B (Dean [2]). *If G is a triangle-free k-connected graph $(k \geq 3)$, then G has an induced cycle C such that every edge of C is k-contractible and that $G - V(C)$ is connected.* ∎

On the other hand, Thomassen [5] proved the existence of a cycle in a k-connected graph whose deletion reduces its connectivity by at most three.

Theorem C (Thomassen [5]). *Let G be a k-connected graph $(k \geq 4)$. Then G has an induced cycle C such that $G - V(C)$ is $(k-3)$-connected and $e_G(v, V(C)) \leq 3$ for all $v \in V(G) - V(C)$, where $e_G(v, V(C))$ is the number of edges joining v and $V(C)$.* ∎

The purpose of this paper is to unify Theorems A and C, and to prove the following theorem.

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Theorem 1. *Let G be a triangle-free k-connected graph ($k \geq 4$). Then G has an induced cycle C such that every edge of C is k-contractible in G and such that $G - V(C)$ is $(k-3)$-connected.*

Note that this theorem also implies Theorem B for $k \geq 4$.

For $e \in E(G)$, we write G/e for the graph obtained from G by contracting e. If $e = xy$, we write v_{xy} for the image of x and y by the contraction. For a k-connected graph G, we write $E_k(G)$ for the set of all k-contractible edges of G. We denote by $d_G(x)$ the degree of a vertex x in G, and by $\Gamma_G(x)$ the set of all vertices adjacent to x in G. For a k-connected graph G, we denote by $\mathfrak{C}_k(G)$ the set of all k-cutsets of G, and by $\mathfrak{C}'_k(G)$ the set of all k-cutsets of G which contain a pair of adjacent vertices. Note that an edge $e = xy \in E(G)$ is not k-contractible if and only if there exists $S \in \mathfrak{C}'_k(G)$ such that $\{x, y\} \subset S$. For $X \subset V(G)$, we denote by $\langle X \rangle_G$ the subgraph of G induced by X. Notation not explained here can be found in [1].

We use the following lemma in the proof of the main theorem.

Lemma 2. *Let G be a triangle-free k-connected graph and $e = xy \in E_k(G)$.*
(1) If $ab \in E_k(G/e)$ and $v_{xy} \neq a, b$, then $ab \in E_k(G)$.
(2) If $v_{xy}a \in E_k(G/e)$, then $xa \in E_k(G)$ or $ya \in E_k(G)$.

Proof. (1) Assume $ab \in E_k(G/e)$ and $ab \notin E_k(G)$. This is possible only if $\{a, b\} \subset S$ and $\{x, y\} \cap S \neq \emptyset$ for some $S \in \mathfrak{C}'_k(G)$, and $\{x, y\} - S$ is a connected component of $G - S$. We may assume $x \in S$ and $y \notin S$. However, this implies $\Gamma_G(y) = S$ and $\{y, a, b\}$ forms a triangle in G. This is a contradiction.

We can prove (2) similarly. Note that in (2) both xa and ya cannot be edges of G, since otherwise $\{x, y, a\}$ forms a triangle of G. ∎

We also use the following theorem concerning the distribution of k-contractible edges. The proof can be found in Dean [2], Egawa et. al. [3] and Mader [4].

Theorem D (Dean [2], Egawa et. al. [3] and Mader [4]). *Let G be a triangle-free k-connected graph. Suppose $\mathfrak{C}'_k(G) \neq \emptyset$. Take $S \in \mathfrak{C}'_k(G)$ and a connected component A of $G - S$ so that $|A|$ is as small as possible. Then for each $x \in A$, every edge incident with x is k-contractible. In particular, $E_k(G) \neq \emptyset$.* ∎

Now we prove Theorem 1. In fact, we prove the following theorem, which is slightly stronger than Theorem 1.

Theorem 3. *Let G be a triangle-free k-connected graph ($k \geq 4$). Then G has an induced cycle C such that*

(1) $E(C) \subset E_k(G)$.
(2) $G - V(C)$ is $(k-3)$-connected, and
(3) $e_G(v, V(C)) \leq 3$ for all $v \in V(G) - V(C)$.

Proof. Assume the theorem does not hold, and let G be a counterexample of the smallest order. By Theorem D, $E_k(G) \neq \emptyset$. We consider two cases.

Case 1. *For some $e = xy \in E_k(G)$, G/e is also triangle-free.*

The argument in this case is almost the same as that in [5]. Let $G' = G/e$. Then G' has an induced cycle C' such that C' satisfies the conditions (1)–(3) in G'. As in [5] the cycle C' in G' is modified to an induced cycle C in G which satisfies (2) and (3). By Lemma 2, it satisfies (1) as well.

Case 2. *G/e is not triangle-free for any $e \in E_k(G)$.*

If all edges of G are k-contractible, then the theorem follows from Theorem C. Hence we may assume $\mathfrak{C}'_k(G) \neq \emptyset$. Take $S \in \mathfrak{C}'_k(G)$ and a connected component A of $G - S$ so that $|A|$ is as small as possible. Then by Theorem D all edges incident with a vertex in A are k-contractible. Since G is triangle-free, it is easy to see $|A| \geq k$. (See [5].) By the assumption of Case 2, for each $e \in E_k(G)$ there exists a cycle C_e of length four which contains e. Since G is triangle-free, C_e is an induced cycle for every $e \in E_k(G)$.

We claim that if a cycle C of length four contains a k-contractible edge, then $G - V(C)$ is $(k-3)$-connected. If $G - V(C)$ is not $(k-3)$-connected, then there exists $T \in \binom{V(G)-V(C)}{k-4}$ such that $G - V(C) - T$ is disconnected. This implies $V(C) \cup T \in \mathfrak{C}'_k(G)$. This is a contradiction since $T \cup V(C)$ contains a k-contractible edge, and thus the claim is proved. In particular, $G - V(C_e)$ is $(k-3)$-connected for each $e \in E_k(G)$.

It is obvious that $e_G(v, V(C_e)) \leq 3$ for all $v \in V(G) - V(C_e)$ since G is triangle-free. Since G is a counterexample, we have $E(C_e) \not\subset E_k(G)$ for all $e \in E_k(G)$.

Take $a_0 \in A$ arbitrarily, and let $\Gamma_G(a_0) \cap A = \{a_1, \ldots, a_m\}$ and $e_i = a_0a_i$ $(1 \leq i \leq m)$. Consider C_{e_i}. Let $C_{e_i} = a_0a_ib_ic_i$. By Theorem D, the edges a_0c_i, a_0a_i, a_ib_i are k-contractible. Thus b_ic_i is not k-contractible and this is possible only if $\{b_i, c_i\} \subset S$.

If $b_i = b_j$ for some i, j, $i \neq j$, then all edges of the cycle $a_0a_ib_ia_j$ are k-contractible by Theorem D and hence it satisfies (1)–(3), a contradiction. Therefore, $b_i \neq b_j$ for all i, j, $1 \leq i < j \leq m$.

We claim that $d_G(a_0) = k$. Let $S_0 = S \cap \Gamma_G(a_0)$. Then $\{b_1, \ldots, b_m\} \cap S_0 = \emptyset$ since G is triangle-free. Thus $|S_0| \leq k - m$. Therefore, $d_G(a_0) = |S_0| + |\Gamma_G(a_0) \cap A| \leq k$. Hence the claim follows since G is k-connected. Since a_0 is chosen arbitrarily, this means that $d_G(v) = k$ for all $v \in A$.

Next, consider a_i. Since G is triangle-free, $S_0 \cap \Gamma_G(a_i) = \emptyset$. By the argument in a previous paragraph $\{b_1, \ldots, b_m\} \cap \Gamma_G(a_i) = \{b_i\}$. Therefore, $|\Gamma_G(a_i) \cap S| = 1$ for all i, $1 \leq i \leq m$. Since a_0 is chosen arbitrarily, this means that $|\Gamma_G(v) \cap S| = 1$ for all $v \in A$. From these observations, it follows that $m = k - 1$ and $c_i = c_j$ for all i, j, $1 \leq i < j \leq k - 1$. Let $b_0 = c_1$.

We now have $S = \{b_0, b_1, \ldots, b_{k-1}\}$ and $S - \{b_0\} \subset \Gamma_G(b_0)$. Arguing in a similar fashion with a_i in place of a_0, we get $S - \{b_i\} \subset \Gamma_G(b_i)$ $(1 \leq i \leq k-1)$. Therefore, $\langle S \rangle_G$ is a complete graph. Since $k \geq 4$, this contradicts the assumption that G is triangle-free. This is the final contradiction in Case 2, and the proof of the theorem is complete. ∎

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